LINEAR AND QUADRATIC GRAPHS WITH THE AID OF TECHNOLOGY

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This paper describes the trial of a unit of work on linear and quadratic graphing with six year 10 classes. Two treatments were developed. The computer treatment made use of the ANUGraph software package, while the calculator treatment paralleled the computer treatment but used a combination of previously prepared graphs and graphs constructed by the student with the aid of a calculator. The emphasis in both treatments was on the interpretation of graphs related to real situations. Comparisons between pre-test and post-test results and interviews with twelve students showed that students learnt to handle the software proficiently, and that both groups improved on most of the topics taught. However, the calculator group seemed to be advantaged by practising plotting of points by hand. Implications for future work are discussed.

Most secondary schools are now equipped with computers that can run graphing and spreadsheet software, although they are still not frequently used for mathematics teaching. In Asp, Dowsey and Stacey (1992) we described a trial of a teaching unit using spreadsheets to assist students in constructing meaning for the important concepts of variable, expression, equation and solution. This teaching experiment gave an indication of which ideas about equations become easier with appropriate computer technology and what difficulties, both conceptually and in terms of classroom management, result from the use of this technology. In this paper we describe results from a trial of a teaching unit on linear and quadratic functions and their graphs, a topic which seems similarly suited to the use of computers.

Frequently all aspects of a complex mathematical idea cannot be expressed with a single representational system. The idea may require multiple, linked representations for its full expression and these different representations may aid the learner's understanding of the idea. Kaput (1992) sees the ability to make translations from one representation of a function to another as a particularly important aspect of mathematical thinking which may be enhanced by technology. The convenient access provided by graphing software to numerical and graphical representations of a variety of functions may assist students to develop a broader and deeper understanding of the function concept. Graphing software enables rapid and automatic translation between algebraic, graphical and numerical (tabular) representations, whereas translation by hand is generally a slow and laborious process for students.

Teaching linear and quadratic graphs using the new technology may bring students to insights that enable them to construct richer and more coherent graphing concepts and develop important techniques needed to reason with mathematical ideas. On the other hand, the use of such technology may encourage students to use ad hoc, intuitive procedures with limited application in the future study of mathematics. In addition, it is possible that the use of computer software for drawing graphs may result in students having insufficient practice on skills such as plotting which may be essential for the development of a good understanding of functions and their graphs. For calculator use in primary schools, these questions seem to have been resolved (see, for example, Hembree and Dessart, 1986) but they now need to be resolved as computing and calculator technology begin to find a place in the algebra program.

To explore aspects of the use of technology in learning secondary mathematics a study based on a unit of work on functions and graphing was conducted with all the Year 10 students and their teachers from a northern Melbourne secondary college. In keeping with what we feel will be the aspects of graphing that remain important when all users of mathematics have free access to graphing software, the unit emphasised using graphs to describe real situations and interpretation of graphs. The study was designed to investigate students' understanding of linear and quadratic functions and their graphs and in particular, sought information on the following questions:

1. What effect does doing less graphing of functions by hand from calculated values have on students' understanding of functions?

2. Given access to graphing software, can students learn to swap between algebraic and graphical representations to solve problems?

In addition, we were interested in questions related to the classroom use of computers such as:

3. What is the 'lead-time' for students to learn how to use graphing software for mathematical purposes?

4. Can students remember how to use software between units of work or must they relearn?

5. What is the most appropriate form of teaching material to guide students' work at the computer?

The 1992 experiment was the first trial for the unit on graphing. In this short report, we will summarise the results which are now informing our further development of the material. In particular, we discuss the results of the pre-test, post-test and interviews which have highlighted skills and concepts which are difficult for students and will need greater attention in future versions of the unit.

PROCEDURE

There were six classes of Year 10 at the school, taught by four teachers. Two parallel treatments for the unit on linear and quadratic functions and their graphs were developed. Each unit was designed to take about ten lessons and replace a textbook unit. The treatments were randomly assigned, subject to the requirements that there were three classes per treatment and the two teachers with two Year 10 classes should have one class assigned to each treatment. Each student was provided with an individual work booklet (two versions) and notes were given to their teachers on appropriate class discussion points. At meetings with the researchers, teachers advised on the directions taken in the booklets and were given an introduction to the computers and the software.

One treatment (the "computer classes") made constant use of ANUGraph in the school's new laboratory of Macintosh computers to create the graphs required. A description of this treatment is given in Asp et al (1992b). For the other treatment, the needed graphs were either plotted by hand using a calculator or were given. As one example, the computer classes found solutions to simultaneous equations by using ANUGraph to draw the graphs of the two functions. The co-ordinates of the intersection points are given automatically when the mouse is appropriately positioned: the scale on the axes is not used. In contrast, the calculator classes were supplied with axes on which the graph of one function was provided and students plotted the other (usually easier) function by hand to find the solutions. This required reading the co-ordinates using the scale. Copies of the booklets used in both treatments are available from the authors.

Written pre- and post-tests for determining students' understanding were administered to all classes. A total of 136 students sat for the pre-test and 132 for the post-test. In addition, at the end of the unit, interviews were conducted with 12 students who were randomly chosen subject to the constraint that there were 2 from each class, one scoring near the 80th and one near the 40th percentile on the pre-test. The six computer treatment students used the computers during their interviews. The purposes of the interviews included examining student thinking, knowledge and facility in using the computers.

RESULTS AND DISCUSSION

A brief discussion of classroom use of the computers and software follows, and then the results are presented under the headings of reading and plotting points and then according to type of translation between representation (graphical, algebraic, numerical, situation). Because of shortage of space, not all questions or all parts of questions on the pre-and post-tests are discussed and only the main findings are illustrated.

Classroom use of computers and software.

Both teachers and students were able to learn how to use the computers confidently in a few lessons. During these early lessons, it was valuable to have the presence of a second adult (the research assistant observing the lessons). When working in the computer laboratory the teachers tended to adopt a management role and seemed to miss important teaching opportunities. This was also observed in our teaching experiment on equation solving with computers (Asp et al, 1992a). This continued despite the provision of notes for teachers on important points for class discussion and urging by the experimenters to spend some time in the computer room not on the machines. ANUGraph performed very satisfactorily. Initially we found that not being able to name the axes was a serious source of confusion. For example, the first lessons were based around a profit-loss context. Students had to work with graphs (profit in dollars vs litres sold) which had axes that were not named. The teachers and research assistant eventually came to believe that having to sort out the confusion made students consider very carefully what quantities were being represented on each axis. Future versions of the unit will suggest that students draw a set of axes on paper and name them before going to the computer to start work on the problem.

Reading and plotting points.
The first question on the pre-and post-tests required students to plot points on a cartesian plane, which we regard as a essential pre-requisite skill to using graphs. The scales on the x-axis and y-axis were deliberately chosen to be different, as frequently happens with graphing software. On the x-axis, labels (numbers) were written at intervals of 0.5 and a grid line was shown at twice this frequency. On the y-axis, labels were written at intervals of 2.0 and a grid line was shown at twice this frequency. The points which students were asked to plot could therefore be on or off a grid point and with labelled or unlabelled x and y co-ordinates. On both the pre-test and post-test, there was a facility of over 85% for plotting labelled grid points in the positive quadrant. For other points, the most common errors were interchanging x and y values, problems with plotting negative values and problems related to scale. Scale problems were linked both to points not being labelled and not being on the grid. For example, the point (2.5,1.5) did not correspond to a grid point; it was frequently plotted as (2.5,1) which was a grid point. Some students were able to plot points only when the corresponding x- and y-axis values were clearly labelled and could not interpolate between them. Similarly, at the interviews, only 7 out of 12 students were able to read off from a graph a y-coordinate (3) which fell halfway between the scale labels 2 and 4; the others gave 2.5 as their answer.

On the pre-test, 42% of students were able to plot all four points correctly, whereas 53% were able to do so on the post-test. To test for significance, the number of changes from pre- to post-test were noted and a simple sign test for the significance of changes was used. This showed no significant change on this question. Further investigation showed that five classes showed improvement whereas the sixth class (a computer class) deteriorated markedly. The deterioration was in fact almost entirely due to a sharp increase in the number of students who interchanged x and y co-ordinates. The reason why this one computer class was so much worse is not clear, but the computer groups had much less practice at plotting points by hand than the calculator groups and at reading the scale (an automatic facility of the software).

The calculator groups also showed more improvement in reading points. Question 7, for example, showed the graphs of a linear and quadratic function $y = x^2 + 2x + 2$ and $y = 3x + 4$ on the one set of axes and their two points of intersection (both of which occurred at grid points and one of which was labelled). Students were asked to give the co-ordinates of the points of intersection. The errors once again involved interchanging the coordinates, scale and negative values. The computer group showed little improvement from pre-test to post-test (41 correct to 45 correct) but the calculator group improved significantly (21 correct to 47 correct), although from a lower base. The sign test for the significance of changes indicates significance at the 5% level ($p = 0.0226$). Results for the second part of question 7 which asked students to solve the equation $x^2 + 2x + 2 = 3x + 4$ were appalling: only one student on the pre-test (by solving the quadratic) and three on the post-test were able to give two correct answers for x! Further, only 24 students on the pre-test and 9 on the post-test were able to give even one correct value for x. Less than half the students attempted this part on either test. Of those that did, the most common source of errors was incorrect algebra.

We had assumed that Year 10 students would have mastered the skills of reading and plotting points. However in the light of the performance of the Year 10 students in this pilot study, future versions of the units need to give more explicit attention to reading and plotting awkward points.

Interpreting graphs: graphs to situations.

Question 5 involved a context leading to a quadratic expression for the area A m^2 of a rectangular enclosure in terms of the width x m; the context was similar but not identical on the pre- and post- tests. The quadratic graph showing how A varied with x was given and students were asked to read a value of A for a value of x, values of x for a value of A and the greatest value of A. All these questions were posed in terms of the context, e.g. "If the area enclosed is 3 m^2, what could the width of the garden be?". A significant proportion of students did not attempt this question (about 30% on the pre-test and 18% on the post-test). On the pre-test, 19% were correct on all three graph interpretation parts; this rose to 33% on the post-test. The sign test for the significance of changes shows there was an improvement significant at the 0.5% level ($p = 0.0046$) in performance overall on the interpretation of the graph.

Once again some students when reading A from x were unable to interpolate and went for the nearest grid point (about 10 instances on both tests). Nevertheless, on this specific item, there was a significant improvement in performance for the calculator group (sign test — $p = 0.0000$), but not for the computer group. In reading x from A, very few students gave two solutions (4 on the pre-test and 8 on the post-test); otherwise the one answer given was usually correct, probably because it corresponded to a grid point. Again the sign test applied to the number of changes from pre- to post-test for students in each group giving at least one correct answer on this item indicated a significant improvement in performance for the calculator group ($p = 0.0000$), but not for the computer group. Finding the maximum value seemed to be an easy aspect of graphical interpretation to teach. In this case the maximum value was a labelled grid point, and the percentage of students correct increased from 20% to 42%. Only the computer group showed statistically significant improvement (sign test — $p = 0.0024$). Many students who attempted to use algebra instead of the graph made errors because their own expression for the area was incorrect. In fact the results were extremely poor in every instance where students used algebra on the pre- and post-tests and at the interviews.

Graph to algebra.

Question 8 showed the graph of a straight line that passed through marked intercepts on the two axes and students were required to give the rule of the straight line (the equations on the two tests were respectively $y = 2x - 4$ and $y = 2x + 4$). On each test, only about two thirds of students gave a response. On the pre-test, 15% were correct and this rose to 17% on the post-test — there was no significant improvement. Further analysis showed that all the improvement was in any case due to one class in which the number of students correct was 3 on the pre-test and 17 on the post-test! (One assumes a highly effective teacher intervention at some stage of the project.) A large proportion of students (24% on the pre-test and 14% on the post-test) gave an equation of some linear appearance but with neither gradient nor y-intercept correct: $y = mx + c$ was one of the better answers. The lessons had not specifically taught students to find the equations corresponding to graphs, but we had expected that the link between algebraic form and the shape of a graph would have been strengthened. This is discussed below.

Algebra to graph.

The difficulties in making the translation from algebra to graphs were evident during the twelve interviews, where the only students to draw the straight line $y = 14 - 2x$ correctly were the three higher ability students from the calculator classes. Many students wanted to take the coefficients in the formula and directly give them a concrete meaning. For example, in attempting to graph $y = 14 - 2x$, four students put the graph through 14 (labelled on the y-axis) and +2 or −2 on the x-axis, while another plotted the single point (2,14). When sketching the quadratic $h = -0.75x^2 + 27x + 2$ in another item about the height h above water of a distress flare x seconds after firing, one student explained the parabolic shape thus: "Because it starts on 0.75 then goes up by 27 then it moves across 2". Another commented: "It goes from -0.75 up to 27". Only four of the twelve indicated some reasonable sort of parabola.

As noted above, calculator students made better attempts at graphing $y = 14 - 2x$. However, the two groups performed similarly on the next question which required them to indicate on graphs they had drawn where $14 - 2x$ is greater than, equal to or less than $3x - 6$ (when students did not successfully draw the graphs, they were given a correct graph to use). The teaching program had used several "real-life" contexts which had required decisions of this nature. Only high ability students were able to go on to use algebra to check when $14 - 2x = 3x - 6$.

In addition to asking students to draw and sketch graphs, we tested to see if they could match algebraic and graphical form. Question 9 gave the graphs of one linear and two quadratic functions on separate sets of axes and listed six algebraic functions. Students were to match the graph with the corresponding function(s). No student got all matchings correct on either test. From pre-test to post-test, students became better at matching algebraic and graphical forms. On the pre-test, there were 47 occasions when a linear expression was matched to a quadratic graph; this occurred 18 times on the post-test. Similarly, 37 students on the pre-test and 8 on the post-test chose a quadratic expression for the linear graph. Once again, many errors arose when students looked at significant numerical values associated with the sketch (e.g. the intercepts with either axis, a maximum etc) and then chose any function that seemed to have these numerical values present in some obvious way.

IMPLICATIONS FOR FUTURE WORK

Significant improvements (although not large as one might hope) were observed for both the computer and calculator groups in most of the topics which the lessons targetted, i.e. interpreting graphs (with intersections, maximum values etc) and matching graph shape and algebraic form. There were few differences in the mathematical gains made by the calculator and computer groups, although the calculator groups made more improvements than the computer groups in reading and plotting points and in drawing graphs. On the other hand, the computer groups gained facility in using the technology. We conclude that for the Year 10 students at the school where the trial was conducted, practising these basic aspects by hand was still essential. It was not sufficient to use the ANUGraph feature to read off the co-ordinates of points or to work with the graphs drawn by the computer.

From this study we know that teachers with little previous experience of computers and with only a few hours of prior experience with ANUGraph can successfully manage the computer-based unit of work which we provided. Students can learn to use the computers and software whilst working on the unit, although they did not progress as far as students who used the parallel non-computer material. We have also established within the calculator lessons a variety of techniques (such as providing graphs and axes for students to add other components to by hand) which enabled the students to concentrate on interpretation of graphs and not become entirely bogged down in plotting.

Access to the laboratory of computers is still difficult in the school where we worked. The computer groups generally worked uninterrupted at the machines and seemed to miss opportunities for group reflection and discussion. This was probably because of the perception that every minute in the computer laboratory was highly valuable and should not be wasted. Our preferred mode would therefore now be a mix of time in and out of the laboratory with some of the hand plotting examples trialled by the calculator groups being used throughout the computer based unit. We believe that the ability to use a computer graphing program is a valuable skill for students to develop and recommend that all students have this opportunity at least by Year 10.

It seems worthwhile noting the extreme difficulty students experienced whenever they attempted to use algebra. Even basic calculator skills caused serious difficulties. For example, of the 9 students who did this question in the interviews, only one correctly evaluated $h = -0.75x^2 + 27x + 2$ when $x = 1$. Not having mastered basic skills makes further learning very problematic.

REFERENCES

Asp, G., Dowsey, J. & Stacey, K. (1992). Technology enriched instruction in year 9 algebra. In B. Southwell, B. Perry & K. Owens (Eds.), *Proceedings of the Fifteenth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 84 -93). University of Western Sydney: MERGA.

Asp, G., Dowsey, J., Hutton, B., McLennan, A. & Stacey, K. (1992a). Technology enriched algebra in year 9. In M. Horne & M. Supple (Eds.), *Mathematics: Meeting the Challenge* (pp. 425 - 432). Melbourne: Mathematical Association of Victoria.

Asp, G., Dowsey, J., Hutton, B., McLennan, A. & Stacey, K. (1992b). Technology enriched algebra in year 10. In M. Horne & M. Supple (Eds.), *Mathematics: Meeting the Challenge* (pp. 433-439). Melbourne: Mathematical Association of Victoria.

Hembree, R. & Dessart, D. J. (1986). Effect of hand-held calculators in precollege mathematics education: a meta-analysis. *Journal for Research in Mathematics Education,* 17(2), 83-99.

Kaput, J. (1992). Technology and Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 515 - 556). New York: MacMillian.

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